NEW INTERACTIONS IN q–SCALAR ELECTRODYNAMICS?

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Abstract

We argue that new interactions could be "switched on" by q–deforming physics. As an example, we discuss q–scalar electrodynamics.

1. Introduction

Recently there has been some interest in q–deformed Minkowski space as either an element of a q–regularisation scheme or as a model for 'first order' Planck scale corrections to the geometry of spacetime. The idea is that by making spacetime slightly non–commutative one might be able to regularise singularities in QFT. Such a q–regularisation scheme is hoped to be more natural than others in the sense that one expects the entire structure of the undeformed theory to generalise such that symmetries become q–symmetries, etc. The big question is, however, whether q–deformed physics could show some novel features. This paper argues that genuine 'q–effects' are indeed to be expected by showing that new interactions are possible in q–deformed scalar electrodynamics.

2. Braided matrices as non–commutative spacetime

A natural candidate for a q–deformed spacetime is the algebra of braided Hermitean 2×2 matrices, which was introduced by S. Majid in [2]. Braided matrices have better covariance properties than 2×2 quantum matrices [5] and also have a central and grouplike element, the *braided determinant* $\underline{\det}$, which gives rise to a q–deformed Minkowski metric. However, the way braided matrices were presented in [2], they did not generalise the additive group structure of Minkowski space as a braided coaddition $\underline{\Delta} x = x \underline{\otimes} 1 + 1 \underline{\otimes} x$, where $\underline{\otimes}$ denotes a *braided tensor product* (an ordinary tensor product would in general not allow for $\underline{\Delta}$ to be an algebra map [3]). See the paper by S. Majid in these proceedings and the references therein for an introduction to the theory of braided matrices and braided groups.

In [8] we found such a braided tensor product which allows for a braided coaddition on the algebra braided Hermitean matrices. It was given in terms of a background quantum group (= symmetry group of the q–deformed system) to play the rôle of quantum Lorentz group. The coaction by this quantum group preserves the entire structure of quantum Minkowski space, i.e. both its braided coaddition and its non–commutative algebra structure. See [7] for a comparison with the approach of [1]. The final result is given in terms of two solutions of the four–dimensional quantum Yang–Baxter equation (QYBE):

$$\mathbf{R}'^{ab}_{\ cd} = R^{-1}{}^{LC}_{BI} R^{B'I}_{JA} R^{A'J}_{KD'} \widetilde{R}^{KD}_{C'L}, \qquad \mathbf{R}_+{}^{ab}{}_{cd} = R^{CI}_{JB} R^{B'J}_{KA} R^{A'K}_{LD'} \widetilde{R}^{LD}_{C'I}.$$

Here R is the standard $SU_q(2)$ R–matrix and $\widetilde{R}$ is defined as $((R^{t_2})^{-1})^{t_2}$, where t_2 denotes transposition in the second tensor component. We use multi–indices $a = (AA')$ with $A, A' = 1, 2$. The two R–matrices obey the relation

$$0 = (P\mathbf{R}_+ + 1)(P\mathbf{R}' - 1) = (P\mathbf{R}' - 1)(P\mathbf{R}_+ + 1), \tag{1}$$

and two mixed QYBEs which ensure that quantum Minkowski space can be equipped with a braided coaddition [3]. Here P denotes the permutation matrix. In terms of these data, q–Minkowski space M_q is given as the algebra of quantum covectors $M_q = V'(\mathbf{R}')$ with generators x_a in the notation of [5]. It can be equipped with a star structure $\overline{x}_a = x_{\bar{a}}$, where $\bar{a} = (A'A)$ is the twisted multi–index (We denote the star structure by a bar in order to avoid confusion with the Hodge star operator). Furthermore, the braided determinant in M_q can be written in terms of the q–deformed metric

$$g^{ab} = \varepsilon_{AC} R^{A'C}_{DB} \varepsilon^{DB'}$$

as $\underline{det} = x_a x_b g^{ab}$, where ε_{AB} is the $SL_q(2, \mathbf{C})$–spinor metric.

If we denote the generators of the FRT algebra $A(\mathbf{R}_+)$ by $\lambda^a_{\ b}$, then the quantum Lorentz group $\mathcal{L}_q$ is defined as the quotient of $A(\mathbf{R}_+)$ by the metric relation $\lambda^a{}_c \lambda^b{}_d g^{cd} = g^{ab}$. The q–metric can be used to raise and lower indices of $\mathcal{L}_q$–tensors, i.e. there is a q–metric induced comodule isomorphism. There is also a q–spinor calculus of $\mathcal{L}_q$–tensors and a generalisation of the familiar decomposition of Lorentz transformations into boosts and rotations [8]. However, in order to obtain full covariance of M_q under the coaction by the q–Lorentz group, we have to adjoin $\mathcal{L}_q$ slightly by a single invertible central and grouplike element ς [3, 8]. The extended q–Lorentz group is denoted by $\widetilde{\mathcal{L}}_q$, and its covariant right coaction on q–Minkowski space is given by $\beta : x_a \mapsto x_b \otimes \lambda^b{}_a \varsigma$. Since the element ς measures the degree of elements of M_q, it is often called *dilaton element* [10, 3]. The generators of $\mathcal{L}_q$ also obey a 'determinant relation' given in terms of a q–deformed ε_{abcd} which has non–zero entries

$\varepsilon_{1234} = 1$	$\varepsilon_{1243} = -q^{-2}$	$\varepsilon_{1324} = -1$	$\varepsilon_{1342} = q^2$	$\varepsilon_{1414} = 1-q^2$
$\varepsilon_{1423} = 1$	$\varepsilon_{1432} = -1$	$\varepsilon_{1444} = 1-q^{-2}$	$\varepsilon_{2134} = -1$	$\varepsilon_{2143} = q^{-2}$
$\varepsilon_{2314} = 1$	$\varepsilon_{2341} = -1$	$\varepsilon_{2413} = -q^{-2}$	$\varepsilon_{2431} = q^{-2}$	$\varepsilon_{2434} = q^{-2}-1$
$\varepsilon_{3124} = 1$	$\varepsilon_{3142} = -q^2$	$\varepsilon_{3214} = -1$	$\varepsilon_{3241} = 1$	$\varepsilon_{3412} = q^2$
$\varepsilon_{3421} = -q^2$	$\varepsilon_{3424} = 1-q^2$	$\varepsilon_{4123} = -1$	$\varepsilon_{4132} = 1$	$\varepsilon_{4141} = q^2-1$
$\varepsilon_{4144} = q^{-2}-1$	$\varepsilon_{4213} = 1$	$\varepsilon_{4231} = -1$	$\varepsilon_{4243} = 1-q^{-2}$	$\varepsilon_{4312} = -1$
$\varepsilon_{4321} = 1$	$\varepsilon_{4342} = q^2-1$	$\varepsilon_{4414} = 1-q^{-2}$	$\varepsilon_{4441} = q^{-2}-1$	

3. Forms on quantum Minkowski space

For q–Lorentz tensors, we shall need a q–deformed notion of symmetry and antisymmetry. The relation (1) between the two R–matrices suggests to call an $\widetilde{\mathcal{L}}_q$–tensor $w_{\dots ab\dots}$ *q–antisymmetric* or *q–symmetric* in the adjacent indices a and b if

$$w_{\dots ab\dots} = -w_{\dots cd\dots}\mathbf{R}_{+}{}^{dc}{}_{ab}, \quad \text{or} \quad w_{\dots ab\dots} = w_{\dots cd\dots}\mathbf{R}'^{dc}{}_{ab}, \tag{2}$$

respectively. Using the metric–induced isomorphism mentioned above, we can raise indices and obtain corresponding formulae for upper indices. The matrix ε_{abcd} is q–antisymmetric in any two adjacent indices in the sense of 2, and it is the unique $\mathcal{L}_q$–tensor of valence four with this property. It can thus be used to define the *q–antisymmetrisation* of an $\widetilde{\mathcal{L}}_q$–tensor in the adjacent indices $a_1 \dots a_k$ as

$$w_{\dots[a_1\dots a_k]\dots} = n_k^{-1} w_{\dots b_1\dots b_n\dots}\varepsilon^{c_1\dots c_{4-k}b_1\dots b_k}\varepsilon_{c_{4-k}\dots c_1 a_1\dots a_k} \tag{3}$$

for $n < 5$ and zero otherwise. The normalisation factors n_k are q–deformations of $(4-k)!k!$ [9]. Due to the properties of ε_{abcd}, the q–antisymmetrisation of a tensor is q–antisymmetric in the sense of (2), but it also has the other properties one might expect. For example, all q–antisymmetrisers are projectors, and the zero– and one–dimensional ones are trivial. Furthermore, q–symmetry and q–antisymmetry, which are defined in terms of two different R–matrices, are compatible: if an $\widetilde{\mathcal{L}}_q$–tensor is q–symmetric in any two adjacent indices then its q–antisymmetrisation is zero. The q–antisymmetrisation operation is also $\widetilde{\mathcal{L}}_q$–covariant in the sense that the coaction by the q–Lorentz group commutes with the q–antisymmetrisation.

Using the covariant q–antisymmetrisation for $\widetilde{\mathcal{L}}_q$–tensors, we define *p–forms* $\mathbf{w}$ on q–Minkowski space in our dual and q–deformed setting as $\widetilde{\mathcal{L}}_q$–comodule $*$–morphisms $\mathbf{w} : \Lambda_p \to M_q$, where Λ_p is the complex vector space spanned by the elements $\mathrm{e}_{a_1\dots a_p} = x_{[a_1} \otimes \dots \otimes x_{a_p]}$ of $M_q^{\otimes^p}$ and their conjugates $\overline{\mathrm{e}}_{a_1\dots a_p}$ from a second copy. One can show that all p–forms on quantum Minkowski space are of the form $\mathbf{w}(\mathrm{e}_{a_1\dots a_p}) = w_{[a_1\dots a_p]}$, for some $w_{a_1\dots a_p} \in M_q$. In the following we shall always denote forms by boldface letters and their values on the generators by plain font letters. The collection of all p–forms form a complex vector space Ω_p, with $dim(\Omega_p) = 0$ for $p > 4$. As in the classical case, the space of forms on quantum Minkowski space $\Omega = \bigoplus_{p=0}^{4} \Omega_p$ can be equipped with an algebra structure: Given a p–form $\mathbf{w}$ and an r–form $\mathbf{v}$, their *q–wedge product* $\mathbf{w} \wedge \mathbf{v} \in \Omega_{p+r}$ is defined as:

$$\mathbf{w} \wedge \mathbf{v} : \mathrm{e}_{a_1\dots a_{p+r}} \mapsto w_{[a_1\dots a_p} v_{a_{p+1}\dots a_{p+r}]}.$$

This bilinear operation '$\wedge$' defines an associative algebra structure on Ω, the *q–exterior algebra*, with identity $\mathbf{1} \in \Omega_0$ given by $\mathbf{1} : \mathrm{e}, \overline{\mathrm{e}} \mapsto 1$.

In order to introduce the differential calculus on Ω note that since quantum Minkowski space has a braided coaddition, we can immediately apply the results of [4], where a braided differential calculus was developed. The key idea was to obtain braided differential operators ∂_a by 'differentiating' the braided coaddition. The resulting algebra obeys the relations of $V'(\mathbf{R}')$ and acts in an $\widetilde{\mathcal{L}}_q$–covariant fashion on quantum Minkowski space such that a *braided Leibnitz rule* holds [4]. In terms of these operators, the *q–exterior derivative* $d : \Omega_p \mapsto \Omega_{p+1}$ is defined as

$$d\mathbf{w}(\mathbf{e}_{a_1\ldots a_{p+1}}) = \partial_{[a_1} w_{a_2\ldots a_{p+1}]},$$

and is known to obey $d^2 = 0$ [9]. On wedge products it acts as $d\mathbf{w} \wedge \mathbf{v} = (d\mathbf{w}) \wedge \mathbf{v} + (-1)^p \mathbf{w} \wedge d\mathbf{v}$, where $\mathbf{w}$ is a p–form. We also define a *q–Hodge star operator* $* : \Omega_p \to \Omega_{4-p}$ in terms of the metric g^{ab} and the tensor ε_{abcd} as

$$^*\mathbf{w}(\mathbf{e}_{a_1\ldots a_{4-p}}) = n_p^{-1/2} \varepsilon_{a_1\ldots a_{4-p} b_1\ldots b_p} g^{b_1 c_p} \ldots g^{b_p c_1} w_{[c_1\ldots c_p]}. \tag{4}$$

Its square on p–forms is given by $* \circ * = (-1)^{p(4-p)}$, as in the classical case. In terms of the q–exterior derivative and the q–Hodge star operator, the *q–coderivative* $\delta : \Omega_p \to \Omega_{p-1}$ and the *q–Laplace–Beltrami operator* $\boldsymbol{\Delta} : \Omega_p \to \Omega_p$ on p–forms on quantum Minkowski space are defined as $\delta = {}^*d^*$ and $\boldsymbol{\Delta} = \delta d + d\delta$, respectively. Due to the well–behaved properties of d and the q–Hodge star operator, one also finds $\delta^2 = 0$ for the q–coderivative.

4. Scalar electrodynamics on q–Minkowski space

We have seen that there is a q–deformed exterior calculus on the q–exterior algebra which generalises the classical case in a straightforward fashion. It therefore seems natural to define the *q–d'Alembert equation* and *q–Maxwell equation* as

$$\boldsymbol{\Delta}\varphi = 0, \qquad \delta d\mathbf{A} = 0 \tag{5}$$

respectively, where φ is a zero–form and $\mathbf{A}$ a one–form on quantum Minkowski space. These equations are equivalent to $\Box\varphi = 0$ and $\Box A_z - \partial_z \partial^c A_c = \partial^c \partial_{[c} A_{z]} = 0$, respectively, where '$\partial$' denotes the braided differential operators on M_q, $\Box = \partial_a \partial_b g^{ab}$ the *q–d'Alembert operator* and '[ ]' the q–antisymmetrisation from (3). In this form one can easily see that both equations are $\widetilde{\mathcal{L}}_q$–covariant in the sense that the action of the differential operator commutes with the coaction by the quantum Lorentz group. In [9], we also found families of q–deformed plane wave solutions to these equations and gave a $SL_q(2, \mathbf{C})$–spinor analysis of the self–dual and anti–self–dual parts of the q–field strength tensor $\mathbf{F} = d\mathbf{A}$ with respect to the q–Hodge operator (4).

Similar to the classical case, solutions φ of the q–d'Alembert equation determine a conserved current one–form $\mathbf{j}$

$$\mathbf{j} = \overline{\varphi} \wedge (d\varphi) - q^{-2}(d\overline{\varphi}) \wedge \varphi, \qquad \delta\mathbf{j} = 0 \tag{6}$$

But this current is not necessarily zero for real solutions, i.e. zero–forms φ such that $\overline{\varphi} = \varphi$. For example if a real solution φ commutes with $\partial\varphi$, then $j_a = (1 - q^{-2})\varphi\partial_a\varphi$ does not vanish.

Thus it seems possible to couple a *real* q–scalar field to the q–electromagnetic field. In order to explore this possibility further, note that solutions to the q–Maxwell equation have a *gauge freedom*: If $\mathbf{A}$ is a solution of the q–Maxwell equation and

Λ a zero-form then $\mathbf{A} + d\Lambda$ is also a solution. Provided it is possible to solve the inhomogeneous equation $\Delta\Lambda = -\delta\mathbf{A}$, we can use this gauge freedom to arrange for $\mathbf{A}$ to satisfy the *q-Lorentz gauge* condition $\delta\mathbf{A} = 0$, which can be shown to be satisfied iff $\partial^c A_c = 0$. In this case $\mathbf{A}$ satisfies $\Delta\mathbf{A} = 0$, which is equivalent to $\Box A_z = 0$. A field $\mathbf{A}$ satisfying the q-Lorentz gauge has a *residual gauge freedom* $\mathbf{A} \mapsto \mathbf{A} + d\Lambda$, where Λ is now a solution of the q-d'Alembert equation.

Using this gauge freedom of the q-Maxwell equation, we can now couple the conserved current (6) to the q-electromagnetic field in a gauge covariant fashion. We achieve this by substituting the covariant derivative $D_a = \partial_a + ieA_a$ for the braided differential operators on M_q, where $e \in \mathbf{R}$ is a coupling constant. As an $\widetilde{\mathcal{L}}_q$-covariant interacting model, we then obtain

$$D_a D_b g^{ab} \varphi = 0, \qquad \delta d\mathbf{A} = \mathbf{J}, \tag{7}$$

with $\mathbf{J}$ given by

$$J_a = \overline{\varphi} \wedge (D_a \varphi) - q^{-2}(D_a \overline{\varphi}) \wedge \varphi.$$

As a consistency condition for (7) one finds $\delta\mathbf{J} = 0$, and by setting the coupling constant e equal to zero, one recovers the free case (5). This interacting model is covariant under gauge transformations

$$\varphi \to e^{i\Lambda}\varphi,$$

$$A_a \to A_a - \partial_a \Lambda,$$

where Λ is a real and central element in M_q. Under these gauge transformations the covariant derivative transforms as

$$D_a \to e^{i\Lambda} D_a e^{-i\Lambda}$$

and the current J_a is invariant.

The equations (7) were obtained by closely following the classical minimal coupling procedure, but show two novel features: Firstly, a real q-scalar field φ does not decouple from the q-electromagnetic field, but determines a non-vanishing current and thus a non-zero q-electromagnetic field $\mathbf{A}$. It has a 'charge' of the order $(1 - q^{-2})e$. Secondly, the q-electromagnetic field contributes to the conserved current a factor roughly equal to $(1 - q^{-2})eA_a\overline{\varphi}\varphi$. With some optimism, one could think of this term as describing vertices with two q-scalar and two q-photon legs in a q-deformed quantum field theory.

It therefore seems that a q-deformation of physics could 'switch on' new interactions which are not present in the commutative case. This would be a genuine q-effect and could result in physics on a non-commutative spacetime being quite different from the classical case. A q-deformation of Minkowski space may have more radical effects than just removing some divergences.

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